4.3 – Spanning Sets

$\vec{e}_1, \vec{e}_2 \in \mathbb{R}^3$

$(1,0,0), (0,1,0)$ $\quad (2,3,0)$

Due Sun

Definition: (generalizes linear combination from Section 3.1)
If **w** is a vector in a vector space V, then **w** is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ in V if **w** can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r$, where each k_i is a scalar. The scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, we have $\mathbf{w} = k_1\mathbf{v}_1$, in which case the linear combination is a scalar multiple of the vector.

#2 Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

a. $(-9, -7, -15)$
b. $(6, 11, 6)$
c. $(0, 0, 0)$

$$a\vec{u}+b\vec{v}+c\vec{w} = a\begin{bmatrix}2\\1\\4\end{bmatrix}+b\begin{bmatrix}1\\-1\\3\end{bmatrix}+c\begin{bmatrix}3\\2\\5\end{bmatrix}=\begin{bmatrix}-9\\-7\\-15\end{bmatrix}$$

a) yields $\begin{bmatrix}2&1&3\\1&-1&2\\4&3&5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-9\\-7\\-15\end{bmatrix}$

$$\left[\begin{array}{ccc|c}2&1&3&-9\\1&-1&2&-7\\4&3&5&-15\end{array}\right]$$ Instead of doing this repeatedly,

$$\left[\begin{array}{ccc|c}2&1&3&b_1\\1&-1&2&b_2\\4&3&5&b_3\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&-\frac{11}{2}b_1+2b_2+\frac{5}{2}b_3\\0&1&0&\frac{3}{2}b_1-b_2-\frac{1}{2}b_3\\0&0&1&\frac{7}{2}b_1-b_2-\frac{3}{2}b_3\end{array}\right]$$

Since there are no restrictions on b_1, b_2, b_3, the vectors $\vec{u}, \vec{v}$ & $\vec{w}$ span $\mathbb{R}^3$.

In particular, if $b_1 = -9$, $b_2 = -7$, $b_3 = -15$, then for $a\vec{u} + b\vec{v} + c\vec{w} = (-9, -7, -15)$,

$a = -2$, $b = 1$, $c = -2$ (we can check this)

Appropriate values for b_1, b_2, b_3 provide solutions to parts (b) & (c).

Definition: The **span** of a nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ of vectors is the set of all possible linear combinations of vectors in S, denoted by span $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ or span(S). If $W = \text{span}(S)$, then we say that W is **spanned** by S.

#9 Determine whether the following polynomials span P_2.
$\mathbf{p}_1 = 1 - x + 2x^2$, $\mathbf{p}_2 = 3 + x$, $\mathbf{p}_3 = 5 - x + 4x^2$, $\mathbf{p}_4 = -2 - 2x + 2x^2$

That is, can we build every polynomial of deg ≤ 2 using $\vec{p}_1, \vec{p}_2, \vec{p}_3, \& \vec{p}_4$?

Consider $\vec{p} = a_0 + a_1 x + a_2 x^2$.

Want $a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 + d\vec{p}_4 = \vec{p}$.

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & a_0 - 3a_1 - 2a_2 \end{array}\right]$$

R_3 restricts a_0, a_1, a_2

The set $\{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4\}$ does not span P_2.

Example: (12) Let $T_A : R^2 \to R^2$ be multiplication by A. Determine whether the vector $\mathbf{u} = (1,\ 2)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

a. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

a. $T_A(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T_A(\vec{e}_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Can we form a linear combination of these to obtain $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ if $a=-3, b=2$

yes, $\vec{u} \in \text{span}\{T_A(\vec{e}_1), T_A(\vec{e}_2)\}$

b. $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has no solution

so $\vec{u} \notin \text{span}\{T_A(\vec{e}_1), T_A(\vec{e}_2)\}$

#16 Let W be the solution space to the system $A\mathbf{x} = \mathbf{0}$. Determine whether the set $\{\mathbf{u},\ \mathbf{v}\}$ spans W.

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 3 & -3 & 3 \end{bmatrix} \end{array} \Rightarrow x_2 = x_3 - x_4$$

x_2 is leading

x_1, x_3, x_4 are free

a. $\mathbf{u} = (1,\ 1,\ 1,\ 0)$, $\mathbf{v} = (0,\ -1,\ 0,\ 1)$

b. $\mathbf{u} = (0,\ 1,\ 1,\ 0)$, $\mathbf{v} = (1,\ 0,\ 1,\ 1)$

Let $r = x_1$, $s = x_3$, $t = x_4$

then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$

The solution space is span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

a. $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ No, because we can't create, for instance $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ using $\vec{u}$ & $\vec{v}$.

Similarly we'd find for b, no.

Theorem 4.3.1 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:
a) The set W of all possible linear combinations of the vectors in S is a subspace of V.
b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

pf (a): clearly, $\vec{0} \in \text{span}(S)$. Let $\vec{u}, \vec{v} \in \text{span}(S)$.
Then $\vec{u} = \sum_{i=1}^{r} c_i \vec{w}_i$ and $\vec{v} = \sum_{i=1}^{r} k_i \vec{w}_i$.

So $\vec{u} + \vec{v} = \sum_{i=1}^{r} c_i \vec{w}_i + \sum_{i=1}^{r} k_i \vec{w}_i = \sum_{i=1}^{r} (c_i + k_i) \vec{w}_i$

$\Rightarrow \vec{u} + \vec{v} \in \text{span}(S)$.
Let $k \in \mathbb{R}$. Then $k\vec{u} = k \sum_{i=1}^{r} c_i \vec{w}_i$

$= \sum_{i=1}^{r} (k c_i) \vec{w}_i \Rightarrow k\vec{u} \in \text{span}(S)$
Thus span(S) is a subspace.

Theorem 4.3.2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

#20 Let $\mathbf{v}_1 = (1,\ 6,\ 4)$, $\mathbf{v}_2 = (2,\ 4,\ -1)$, $\mathbf{v}_3 = (-1,\ 2,\ 5)$, and $\mathbf{w}_1 = (1,\ -2,\ -5)$, $\mathbf{w}_2 = (0,\ 8,\ 9)$. Use Theorem 4.3.2 to show that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

Can we use $\vec{v}_i$ to make each $\vec{w}_i$?

$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w}_1 & \vec{w}_2 \end{array}$$
$$\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

yes. $-1\vec{v}_1 + 1\vec{v}_2 = \vec{w}_1$, $2\vec{v}_1 - \vec{v}_2 = \vec{w}_2$

Can we use $\vec{w}_i$ to make $\vec{v}_i$?

$$\begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$
$$\left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, yes.

$$\text{span}\{\vec{v}_i\} = \text{span}\{\vec{w}_i\}.$$